

Solving the General Truncated Moment Problem by the r -Generalized Fibonacci Sequences Method

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We give in this paper a new method for solving the generalized truncated power moment problem. To this aim we use r -generalized Fibonacci sequences and their associated minimal polynomials. We provide an algorithm of construction of solutions in a short method. This method allows us to avoid any appeal to Hankel matrices or any positive definiteness conditions as in the Flessas–Burton–Whitehead (FBW) approach. Examples and general cases are discussed. © 2001 Academic Press

1. INTRODUCTION

The power moment problem (*PMP* for short) has been largely studied in the literature; it arises in various fields of analysis (see [1, 3–6, 9], for example). Recently, it has been shown that the *PMP* has many useful applications in physics. Particularly, the mathematical results and techniques of the *PMP* and generalized moment problem (defined below in this section) can be applied successfully in the theory of renormalized or effective iteration in quantum many-body physics (see [10]), in nuclear physics, and in



solid state physics (see [17–19]). More recently, it appears that the *PMP* is closely related to the maximal entropy (see [13, 14, 16]).

The classical K – power moment problem, K – *PMP*, consists of finding a nonnegative measure μ supported by K such that

$$\int_K t^k d\mu = s_k \quad \text{for every } k \geq 0, \quad (1.1)$$

where $S = (s_k)_{k \geq 0}$ is a given sequence of real numbers and K a closed subset of \mathbb{R} . For $K = \mathbb{R}$ the K – *PMP*, called the *Hamburger PMP*, the existence of solutions of (1.1) is closely related to the positivity of the *Hankel Matrices*,

$$Q_n = \begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & & & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{pmatrix}, \quad (1.2)$$

for every $n \geq 0$ (see [1, 4]). For $K = [a, b]$ or $[0, +\infty[$ the K – *PMP* is called the *Hausdorff* and *Stieltjes PMP*, respectively. These two problems have been studied by several authors using different mathematical approaches (see [1, 4–6], for example).

If $(s_k)_{0 \leq k \leq N-1}$ is a finite sequence of real numbers and $K \subset \mathbb{R}$, the *truncated power moment problem*, K – *TPMP*, consists of finding a nonnegative measure μ (supported by K) such that

$$\int_K t^k d\mu = s_k \quad \text{for every } 0 \leq k \leq N-1. \quad (1.3)$$

The *generalized K – power moment problem* (K – *GMP*) consists of finding a measure μ (supported by K), which is *not necessarily nonnegative*, such that

$$\int_K t^k d\mu = s_k \quad \text{for every } k \geq 0.$$

In this case we say that μ is a *generating measure* of $S = (s_k)_{k \geq 0}$.

The *truncated general moment problem*, K – *TGMP*, is similarly stated.

Without loss of generality, we suppose in this paper that K is a compact subset of \mathbb{R} , and for simplicity we omit any reference to K in the sequel.

Motivated by investigations in many-body quantum physics, Flessas *et al.* gave in [10, 11] an explicit solution to the *TGMP* (see also [17–19]). The authors extended *PMP* techniques to the *GMP* and *TGMP* cases. More precisely, they split the *GMP* in two classical *PMPs*, by considering certain decomposition of the quadratic form related to the *Hankel matrices* associated with (1.2).

The r –generalized Fibonacci sequences (r – *GFS*) play an important role in many fields of mathematics, physics, and computer science. Particularly,

in physics there are some recent applications in order entropy, fractal morphology, Ising model, and optical generation (see [2, 12, 15, 20], for example).

In this paper, we show that for any finite sequence $S = (s_k)_{0 \leq k \leq N-1}$, there exists a solution to the *TGMP* for arbitrary compact $K \subset \mathbb{R}$. To this aim, we consider r -generalized Fibonacci sequences and the associated recursive power moment problem. The basic tools are from [3, 8, 9]. Our method allows us to recover in a simple way the results of [10] (and [11]) without appealing to the positivity of Hankel matrices. In fact, for the classical *PMP* the positivity of the Hankel matrices is equivalent to the positivity of the solution μ (see [1, 4, 5], for example). Without the positivity requirement of μ , there is always a solution, in fact infinitely many (see Section 4). By [3, 8] the treatment of the *TGMP* is reduced to the solution of some Vandermonde systems.

This paper is organized as follows. We present in Section 2 a sketch of the *FBW* method given in [10, 11] to solve the problem. We devote Section 3 to the introduction of r -*GFS* and the associated *GMP*. We show that for such sequences, the *TGMP* and *GMP* are equivalent. We also give necessary and sufficient conditions for r -*GFS* to be a moment sequence. Section 4 is devoted to the explicit construction of the solution of the *GMP* for arbitrary compact set K . We conclude in the last section with some examples and remarks.

2. FBW APPROACH FOR THE GENERALIZED MOMENT PROBLEM

In this section we give a sketch of the Flessas–Burton–Whitehead (*FBW* for short) method for solving *TGMP* (see [10, 11]). Let $S = (s_k)_{0 \leq k \leq 2N-2}$ be $2N - 1$ real numbers and suppose that μ is a measure satisfying

$$s_k = \int_K t^k d\mu \quad \text{for every } 0 \leq k \leq 2N - 2. \quad (2.1)$$

Let $\mu = \mu^+ - \mu^-$ be the Hahn–Jordan decomposition of μ and write

$$\begin{aligned} s_k &= \int_K t^k d\mu = \int_K t^k d\mu^+ - \int_K t^k d\mu^- \\ &= s_k^+ - s_k^- \quad \text{for every } 0 \leq k \leq 2N - 2, \end{aligned} \quad (2.2)$$

where s_k^+ and s_k^- are the moments of μ^+ and μ^- , respectively. The quadratic form Q associated with the problem (2.1) can be written in the form

$$Q = \sum_{i,k=0}^{N-1} s_{i+k} x_i x_k = \sum_{i,k=0}^{N-1} s_{i+k}^+ x_i x_k - \sum_{i,k=0}^{N-1} s_{i+k}^- x_i x_k = Q_1 - Q_2, \quad (2.3)$$

where $Q_1 = \sum_{i,k=0}^{N-1} s_{i+k}^+ x_i x_k$ and $Q_2 = \sum_{i,k=0}^{N-1} s_{i+k}^- x_i x_k$ are the positive quadratic forms associated with μ^+ and μ^- , respectively. Hence, according to the above discussion, the *TGMP* can be split into two *TMPs*.

Consider now \mathcal{A} the linear operator associated with Q by $\langle v_i | \mathcal{A} | v_k \rangle = s_{i+k}$ in some orthogonal basis $(|v_i\rangle)_i$. The canonical form of Q in the basis $(|e_i\rangle)_i$ of eigenvectors is given by

$$Q = \langle x | \mathcal{A} | x \rangle = \sum_{i=0}^{N-1} \lambda_i (x'_i)^2. \quad (2.4)$$

Define the operators

$$\mathcal{A}^+ |e_i\rangle = \max(0, \lambda) e_i \quad \text{and} \quad \mathcal{A}^- |e_i\rangle = \min(0, \lambda) e_i, \quad (2.5)$$

with associated quadratic forms Q^+ and Q^- . We have $\mathcal{A} = \mathcal{A}^+ - \mathcal{A}^-$ and $Q = Q^+ - Q^-$. Thus $Q^+ \geq 0$ and $Q^- \geq 0$. Set $Q^+ = \langle x | \mathcal{A}^+ | x \rangle = \sum_{i,k=0}^{N-1} s_{ik}^+ x_i x_k$ and $Q^- = \langle x | \mathcal{A}^- | x \rangle = \sum_{i,k=0}^{N-1} s_{ik}^- x_i x_k$ in the basis $(|v_i\rangle)_i$. Coefficients $(s_{ik}^+)_{i,k}$ and $(s_{ik}^-)_{i,k}$ are related to the moment problems associated with nonnegative measures. Let $A = (a_{ik})_{i,k}$ be any symmetric matrix. By observing that

$$Q = Q_1 - Q_2 = (Q_1 - A) - (Q_2 - A), \quad (2.6)$$

the problem is reduced to the question of the existence of some symmetric matrix A such that $(Q_1 - A) > 0$ and $(Q_2 - A) > 0$.

This last problem is then solved affirmatively by an inductive construction of the matrix A , using iterative Hankel determinants. The authors make the explicit construction in the case $n = 4$. For further details see [10, 11].

3. r -GENERALIZED FIBONACCI SEQUENCES AND THE ASSOCIATED MOMENT PROBLEM

3.1. r -Generalized Fibonacci Sequences

Let a_0, a_1, \dots, a_{r-1} ($r \geq 2$) be some fixed real numbers with $a_{r-1} \neq 0$ and consider the sequence $S = (s_k)_{k \geq 0}$ defined by the following linear recurrence relation of order r ,

$$s_{k+1} = a_0 s_k + a_1 s_{k-1} + \dots + a_{r-1} s_{k-r+1}, \quad \text{for all } k \geq r-1, \quad (3.1)$$

where s_0, s_1, \dots, s_{r-1} are specified by the initial conditions. This family of sequences, called r -generalized Fibonacci sequences, is largely studied in the literature (see [7], for example). We shall refer to them in the sequel as r -GFS. The polynomial $P(x) = x^r - a_0 x^{r-1} - \dots - a_{r-2} x - a_{r-1}$ is called a characteristic polynomial associated with $S = (s_k)_{k \geq 0}$ given by (3.1).

An $r - GFS$ can be defined in various ways using different characteristic polynomials as is shown in the following example. Let $S = (s_k)_{k \geq 0}$ with $s_k = k + 1$. Then $S = (s_k)_{k \geq 0}$ may be defined by the following recursive relations,

- $s_{k+1} = s_k + s_{k-1} - s_{k-2}$ for $s_0 = 1, s_1 = 2, s_2 = 3$.
- $s_{k+1} = s_k + \frac{1}{2}(s_{k-1} - s_{k-2} + s_{k-3} - s_{k-4})$ for $s_0 = 1, s_1 = 2, s_2 = 3, s_3 = 4, s_4 = 5$.

Therefore, $P_1(X) = X^3 - X^2 - X + 1$ and $P_2(X) = X^5 - X^4 - \frac{1}{2}X^3 + \frac{1}{2}X^2 - \frac{1}{2}X + \frac{1}{2}$ are two characteristic polynomials associated with $S = (s_k)_{k \geq 0}$.

Let \mathcal{P}_S be the set of characteristic polynomials associated with the sequence $S = (s_k)_{k \geq 0}$.

PROPOSITION 3.1. *For every sequence $S = (s_k)_{k \geq 0}$ given by (3.1), there exists a unique characteristic polynomial $P_S \in \mathcal{P}_S$ with minimal degree. Moreover, every $P \in \mathcal{P}_S$ is a multiple of P_S .*

Proof. For $P \in \mathcal{P}_S$, write $P(X) = \prod_{i=0}^{n-1} (X - \lambda_i)^{k_i}$. By the Binet formula we get

$$s_k = \sum_{i=0}^{n-1} \sum_{j=0}^{k_i} c_{i,j} k^j \lambda_i^k \quad (c_{i,k_i} \neq 0), \quad (3.2)$$

where $c_{i,j}$ are solutions of the following system of n -equations

$$\sum_{i=0}^{n-1} \sum_{j=0}^{k_i} c_{i,j} k^j \lambda_i^k = s_k, \quad k = 0, \dots, n-1.$$

Let $d_i = \max\{j \text{ such that } c_{i,j} \neq 0\}$. Then we have

$$s_k = \sum_{i=0}^{n-1} \sum_{j=0}^{d_i} c_{i,j} k^j \lambda_i^k \quad \text{with } c_{i,d_i} \neq 0. \quad (*)$$

The polynomial $P_S(X) = \prod_{i=1}^{n-1} (X - \lambda_i)^{d_i}$ is a characteristic polynomial for $S = (s_k)_{k \geq 0}$.

On the other hand, if $S = (s_k)_{k \geq 0}$ is a sequence satisfying $(*)$, then for every characteristic polynomial P of S , we have λ_i is a zero for P of order at least d_i . Hence P_S provides a positive answer to the proposition. ■

The polynomial P_S is called *the minimal polynomial associated with $S = (s_k)_{k \geq 0}$* .

As shown in Proposition 3.1, the minimal polynomial is a factor of any characteristic polynomials that depend only on the initial conditions. Therefore, it can be computed by “testing” Eq. (3.1) for a finite number of characteristic polynomial factors.

3.2. Equivalence of the TGMP and GMP for $r - GFS$

Let $S = (s_k)_{k \geq 0}$ be an $r - GFS$ and consider the associated GMP

$$\int_K t^k d\mu = s_k \quad \text{for every } k \geq 0. \quad (3.3)$$

Let s_0, s_1, \dots, s_{r-1} be the initial conditions and let $P_S(X) = X^r - a_0 X^{r-1} - a_1 X^{r-2} - \dots - a_{r-1}$ be its characteristic polynomial. Consider the following TGMP

$$\int_K t^k d\mu = s_k \quad \text{for every } 0 \leq k \leq r-1. \quad (3.4)$$

The linear recurrence relation (3.1) allows us to have the following proposition which is a variant of Lemma 2.2 of [9], for the case of GMP.

PROPOSITION 3.2. *Let $S = (s_k)_{k \geq 0}$ be an $r - GFS$ whose initial conditions and characteristic polynomial are s_0, s_1, \dots, s_{r-1} and $P_S(X) = X^r - a_0 X^{r-1} - a_1 X^{r-2} - \dots - a_{r-1}$, respectively. Then the following are equivalent.*

- (i) *There exists a Borel measure μ , solution of the GMP for $(s_k)_{k \geq 0}$.*
- (ii) *There exists a Borel measure μ , solution of the TGMP for $(s_k)_{0 \leq k \leq r-1}$ with*

$$\text{supp}(\mu) \subset Z(P) := \{x \in \mathbb{R} \text{ such that } P(x) = 0\}.$$

Proof. (ii) \Rightarrow (i). If $s_k = \int_a^b t^k d\mu(t)$ for $0 \leq k \leq r-1$ and $\text{Supp}(\mu) \subset Z(P)$ then

$$s_r = \int_a^b [a_0 t^{r-1} + a_1 t^{r-2} + \dots + a_{r-1}] d\mu(t) = \int_a^b t^r d\mu(t).$$

By induction we obtain $s_k = \int_a^b t^k d\mu(t)$ for any $k \geq r$. Consequently, μ is a solution of the GMP for $(s_k)_{k \geq 0}$.

(i) \Rightarrow (ii). From the Binet formula and (3.3), we derive that $P_*\mu = 0$. Thus μ is a measure supported by $Z(P)$, which also says that μ is a discrete measure (see also the Proof of Theorem 4.1 below). ■

4. CONSTRUCTION OF THE SOLUTION OF THE TGMP

We start with the following theorem which gives the solution of problem (3.3).

THEOREM 4.1. *Let $S = (s_k)_{k \geq 0}$ be an $r - GFS$. Then $(s_k)_{k \geq 0}$ admits a generating measure μ if and only if P_S has distinct real roots. Moreover, $\text{Supp}(\mu) = Z(P_S)$.*

Proof. Set $P_S(X) = \prod_{i=0}^{r-1} (X - \lambda_i)$ with $\lambda_0 < \dots < \lambda_{r-1}$, then the Binet formula implies that

$$s_k = \rho_0 \lambda_0^k + \dots + \rho_{r-1} \lambda_{r-1}^k, \quad \text{for any } k \geq 0, \quad (4.1)$$

where $\rho_0, \dots, \rho_{r-1}$ are nonzero real numbers derived from the following Vandermonde system of r linear equations

$$\rho_0 \lambda_0^k + \dots + \rho_{r-1} \lambda_{r-1}^k = s_k, \quad k = 0, 1, \dots, r-1, \quad (4.2)$$

(see [5], for example). Consider the Borelean measure μ given by $\mu = \sum_{j=0}^{r-1} \rho_j \delta_{\lambda_j}$, on the interval $[\lambda_0, \lambda_{r-1}]$. From expression (3.1) we derive that

$$s_k = \int_{\lambda_0}^{\lambda_{r-1}} t^k d\mu(t), \quad \text{for every } k \geq 0.$$

Conversely, suppose that μ is a generating measure associated with $S = (s_k)_{k \geq 0}$ and $P \in \mathcal{P}_S$. Easy computations show that $P.\mu = 0$, which asserts that $\text{Supp}(\mu) \subset Z(P) := \{\lambda_0, \dots, \lambda_n\}$. Thus μ has the form

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{\lambda_j}, \quad (4.3)$$

where ρ_j are nonzero real numbers. Hence

$$s_k = \int_{\text{Supp}(\mu)} t^k d\mu = \rho_0 \lambda_0^k + \dots + \rho_{r-1} \lambda_{r-1}^k \quad (\rho_i \neq 0).$$

We obtain P_S by setting $P_S(X) = \prod_{i=0}^{r-1} (X - \lambda_i)$, which is the minimal polynomial of $S = (s_k)_{k \geq 0}$. ■

As a consequence of Theorem 4.1 we can formulate the solution of the TGMP as follows.

COROLLARY 4.2. *For any given finite sequence $S = (s_k)_{0 \leq k \leq r-1}$, there exists a solution of the associated TGMP.*

Proof. Set $P(X) = \prod_{i=0}^{r-1} (X - \lambda_i)$, where $\lambda_i \neq 0$ ($0 \leq i \leq r-1$) and $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{r-1}$. Consider $S = (s_k)_{k \geq 0}$ the r -GFS whose initial conditions and characteristic polynomial are s_0, \dots, s_{r-1} and $P(X)$, respectively. Then the solution holds by Theorem 4.1. ■

In (4.3) the coefficients ρ_j ($0 \leq j \leq r-1$) are derived from a Vandermonde system of linear equations that can be solved using usual computations. This allows us to avoid any discussion about positivity of Hankel matrices of Section 2.

If $\lambda_i = 0$ for some i in Corollary 4.2, the order of the recurrence relation (3.1) can be reduced to $r-1$. Note also that the support of the measure

μ is any arbitrary finite set containing r real numbers. This fact has been observed in Section 4 of [10].

Let μ be a nonnegative discrete measure on \mathbb{R} and set $\sigma(x) = \mu(]-\infty, x])$. Then σ is a nondecreasing function whose set of discontinuities is exactly $\text{Supp}(\mu)$. Using r -GFS we are able to reformulate Theorem 4.1 of [10] as follows.

COROLLARY 4.3. *Let $S = (s_k)_{\leq k \leq r-1}$ be any finite sequence. The following are equivalent.*

- (1) *The TGMP admits a solution μ such that $\text{cardinal}(\text{supp}(\mu)) \leq p$;*
- (2) *For every $n \geq p+1$, we have*

$$\begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & & & \vdots \\ s_n & s_1 & \dots & s_{2n} \end{vmatrix} = 0 \quad (4.4)$$

- (3) *There exist a_0, \dots, a_{r-1} real numbers such that*

$$s_{k+1} = a_0 s_k + \dots + a_{p-1} s_{k-p+1} \quad \text{for } p \leq k \leq r-2.$$

If one of the conditions in Corollary 4.3 is satisfied, then S has a completion in an r -GFS which is a moment sequence associated with a discrete measure.

We end this section by giving some examples that show how r -Generalized Fibonacci sequences are used to give solutions to the TGMP with arbitrary support.

EXAMPLES 1. Let $s_0 = 1$ and s_1 be any given numbers and let $(s_k)_{k \geq 0}$ be an 2 -GFS such that $s_{k+1} = a_0 s_k + a_1 s_{k-1}$ and suppose that $P(X) := X^2 - a_0 X - a_1$ has two distinct roots λ_1 and λ_2 . Consider ρ_1, ρ_2 solutions of the system

$$\begin{cases} \rho_1 + \rho_2 &= 1 \\ \rho_1 \lambda_1 + \rho_2 \lambda_2 &= s_1. \end{cases}$$

Then the measure

$$\mu := \rho_1 \delta_{\lambda_1} + \rho_2 \delta_{\lambda_2}$$

is a solution of the general truncated moment problem associated with s_0, s_1 .

EXAMPLES 2. Given $s_0 = 1$, s_1 and s_2 real numbers, consider $(s_k)_{k \geq 0}$ the 3 - GFS defined by the following recursive relation $s_{k+1} = a_0 s_k + a_1 s_{k-1} + a_2 s_{k-2}$ ($k \geq 3$), with

$$P(X) = X^3 - a_0 X^2 - a_1 X - a_2 = (X - \lambda_0)(X - \lambda_1)(X - \lambda_2).$$

Let μ be the measure

$$\mu := \rho_0 \delta_{\lambda_0} + \rho_1 \delta_{\lambda_1} + \rho_2 \delta_{\lambda_2},$$

where ρ_0 , ρ_1 , and ρ_2 are solutions of the system of equations

$$\begin{cases} \rho_0 + \rho_1 + \rho_2 &= 1, \\ \rho_0 \lambda_0 + \rho_1 \lambda_1 + \rho_2 \lambda_2 &= s_1, \\ \rho_0 \lambda_0^2 + \rho_1 \lambda_1^2 + \rho_2 \lambda_2^2 &= s_2. \end{cases}$$

The numbers ρ_0 , ρ_1 , and ρ_2 are given by

$$\begin{cases} \rho_0 = \frac{s_2 - s_1(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)} \\ \rho_1 = \frac{-s_2 + s_1(\lambda_0 + \lambda_2) - \lambda_0 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)} \\ \rho_2 = \frac{s_2 - s_1(\lambda_0 + \lambda_1) + \lambda_0 \lambda_1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)}. \end{cases}$$

It is obvious that μ is a solution of the TGMP associated with $\{s_0 = 1, s_1, s_2\}$.

5. SOME CONCLUDING REMARKS

We give in this section some comments and remarks about the results given throughout this paper.

Referring to Examples 1 and 2 in the last section, we can provide an algorithm to solve the TGMP. Let $S = (s_k)_{0 \leq k \leq r-1}$ be any r real numbers. A solution of the TGMP associated with S is provided by a discrete measure $\mu := \rho_0 \delta_{\lambda_0} + \rho_1 \delta_{\lambda_1} + \dots + \rho_{r-1} \delta_{\lambda_{r-1}}$, where ρ_i ($i = 0, \dots, r-1$) are computed by solving the Vandermonde system of r - equations given by (4.2). Note that the determinant of such a system is $\prod_{i < j} (\lambda_i - \lambda_j) \neq 0$. Hence the solution with $\text{card}(\text{Supp}(\mu)) = r$ is uniquely defined.

The basic tool to prove the existence of solutions of the GMP for some r - GFS is the associated minimal polynomial P_S . As shown in Theorem 4.1 such a solution exists if and only if P_S has distinct real roots. If $P_S(X) = \prod_{i=0}^p (X - \lambda_i)^{p_i}$, then $S = (s_k)_{k \geq 0}$ is a moment sequence for some distribution,

$$T = \sum_{i=0}^p \sum_{j=1}^{p_i} \rho_{i,j} \delta_{\lambda_i}^{(j)},$$

where $\delta_{\lambda_i}^{(j)}$ is the j -derivative in the sense of distributions of the Dirac measure δ_{λ_i} .

Given $S = (s_k)_{0 \leq k \leq r-1}$, according to Corollary 4.3, it is always possible to choose a measure μ such that $\text{card}(\text{supp}(\mu)) = r$. For a finite subset K of \mathbb{R} such that $\text{card}(K) \leq r - 1$ it is possible to find a measure supported by K if and only if condition (2) or (3) of Corollary 4.3 is satisfied. However, it is always possible to find a distribution supported by K interpolating S .

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